

Self-duality and associated parallel or cocalibrated G_2 structures

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Abstract

We prove the existence of a large family of naturally defined G_2 structures on certain compact principal $SO(3)$ -bundles P_+ and P_- associated with any given oriented Riemannian 4-manifold M . A nice surprise is that such structures, though never calibrated, are always cocalibrated. As we start our study with a recast of the Bryant-Salamon construction of G_2 holonomy on the vector bundle of anti-selfdual 2-forms on M , we then discover *incomplete* examples of that restricted holonomy on disk bundles over \mathcal{H}^4 and $\mathcal{H}_{\mathbb{C}}^2$, respectively, the real and complex hyperbolic space. Also, the existence of a G_2 metric on $\Lambda_+^2 T^*M$ for any $K3$ surface M is shown.

Key Words: selfdual metric, calibrated, holonomy

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Introduction

The group G_2 of automorphisms of the octonians is equally characterized as the group of invariants of a certain 3-form $\phi \in \Lambda^3(\mathbb{R}^7)^*$. This Lie subgroup of $SO(7)$ gives birth to a special Riemannian 7-dimensional geometry whose basics are fairly well-known today. Following a thorough study by R. Bryant in [Bry87], such geometry began to have great attention and led to various deep insights and new questions.

Let M be an oriented 4-dimensional Riemannian manifold. The structures we present here are defined on the total space of *two* natural principal $SO(3)$ -bundles P_+ and $P_- \rightarrow M$, abbreviated P_{\pm} , of essentially orthonormal coframe basis $\{e^1, e^2, e^3\}$ of (anti-)selfdual 2-forms on M and invoking the induced connection 1-form $\omega \in \Omega_{P_{\pm}}^1(\mathfrak{o}(3))$ in canonical matrix form. A structure 3-form ϕ on P_{\pm} , say a preferred G_2 structure within the family we found, may be given immediately:

$$\phi = \omega^1 \wedge \omega^2 \wedge \omega^3 \mp (e^1 \wedge \omega^1 + e^2 \wedge \omega^2 + e^3 \wedge \omega^3) .$$

We may indeed challenge the reader in saying that a basic knowledge of the theory, up to Bianchi identity in 4-dimensional geometry, is just enough to prove ϕ is coclosed. Notice our result may be relevant in finding explicitly G_2 cocalibrated metrics. But it is more in revealing a new twistorial approach to 4-manifolds and the existence of new functorial relations. Also, in another perspective, our result compares with the well-known theorem which says that every cotangent bundle is a symplectic manifold.

We start our study with a recall of the theory of connections on principal coframe bundles and the Singer-Thorpe curvature decomposition for Riemannian 4-manifolds. We hope to have given an independent proof of this decomposition. The many well-known results are used frequently along the main theorems. We also present an introduction to G_2 fundamental notions and equations. Then we revisit the G_2 holonomy metrics on $X_{\pm} = \Lambda_{\pm}^2 T^*M$ constructed by R. Bryant and S. Salamon, somehow willing to honour their discovery of true G_2 holonomy. We compute the fundamental torsion equations of [FG82] on X_{\pm} , for M anti-selfdual, or selfdual for the $-$ case, which are finally supremely related by an elementary lemma about two 1-variable dependent positive functions (throughout the paper we work in the smooth category). The torsion forms, also described for the new structures on P_{\pm} , thus entail many new unsolved questions. As our computations are also accomplished for the bundle of self-dual 2-forms, we use results of C. LeBrun to deduce that to every $K3$ surface with Calabi-Yau metric corresponds a 2-parameter family of parallel G_2 structures on $\Lambda_{+}^2 T^*K3$. Our last chapter contains the general equations of the new, always cocalibrated, G_2 structures.

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1 Riemannian 4-manifolds and G_2 7-manifolds

1.1 Frame bundle and connection forms

We start by recalling some classical elements of differential and Riemannian geometry, which may be seen in many references such as [Hel78, KN96]. Introducing notation, we let $\Omega_Y^p(E)$ or $\Omega^p(Y, E)$ be the space of smooth sections $\Gamma(Y; \Lambda^p T^*Y \otimes E)$ if we are given $E \rightarrow Y$ a vector bundle and Y a manifold. Also, we let $\Omega_Y^p = \Omega_Y^p(\mathbb{R})$.

To begin with, we let M be a smooth n -dimensional manifold and let F^*M be the principal $\mathrm{GL}(n, \mathbb{R})$ -bundle of coframes. A coframe $e \in F^*M$ is a linear isomorphism $(e^1, \dots, e^n) : T_m M \rightarrow \mathbb{R}^n$. The involved Lie group right-action $g \mapsto R_g(e) = e \cdot g$ is defined by $e \cdot g = (\sum_j e^j g_j^1, \dots, \sum_j e^j g_j^n)$, $\forall g \in \mathrm{GL}(n, \mathbb{R})$.

Using the bundle projection $\pi : F^*M \rightarrow M$ we have a canonical \mathbb{R}^n -valued 1-form θ on F^*M , the so-called *soldering* form. It is one which gives a first example of a tautological form, as it is defined by

$$\theta_e = e \circ \pi_* . \quad (1)$$

Now suppose the manifold is endowed with a linear connection, that is, essentially a covariant derivative or a first-order operator commuting with restrictions and satisfying Leibniz rule. Given any local section $s = (e^1, \dots, e^n) : U \rightarrow F^*M$ on an open subset $U \subset M$, we then have a matrix valued 1-form ω given by the connection: $\nabla e^i = \sum_j e^j \otimes \omega_j^i = s \omega^i$. A natural extension d^{∇} of ∇ as a differential operator on the

relevant space leads us to the *curvature* tensor $R^\nabla = (d^\nabla)^2$, and locally to a *curvature* form ρ_k^i ; respectively a T^*M -valued 2-form on M

$$R_{Z_1, Z_2}^\nabla e^i = \nabla_{Z_1} \nabla_{Z_2} e^i - \nabla_{Z_2} \nabla_{Z_1} e^i - \nabla_{[Z_1, Z_2]} e^i, \quad \forall Z_1, Z_2 \in TM, \quad (2)$$

and a Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ -valued 2-form on U

$$\rho_k^i = d\omega_k^i + \sum_j \omega_k^j \wedge \omega_j^i. \quad (3)$$

Of course, (2) and (3) are related by $R^\nabla e^i = s\rho^i$ and differentiating again gives the so-called Bianchi identity.

More important here, with a statement we cannot prove easily, is that the connection can be completely described over the manifold F^*M . Indeed, there exists a globally defined, unique $\omega \in \Omega^1(F^*M, \mathfrak{gl}(n, \mathbb{R}))$ such that

$$\nabla s = s s^* \omega, \quad \forall s \in \Omega^0(U, F^*M), \quad (4)$$

and such that, for any fundamental vertical vector field $V_e \in TF^*M$, $e \in F^*M$,

$$\omega(V_e) = V \quad (\text{by definition } V_e = \frac{d}{dt} e \cdot \exp(tV), \quad V \in \mathfrak{gl}(n, \mathbb{R})). \quad (5)$$

From this and the existence of time-dependent parallel sections we have that $H = \ker \omega$ is complementary to the vertical tangent subspace, i.e. $\ker \pi_* \subset TFM$. It follows easily that $R_g^* \omega = \text{Ad}(g^{-1})\omega$, $\forall g \in \text{GL}(n, \mathbb{R})$, most of this being common to *other* linear connections.

The following are two famous equations of Cartan, fundamental for the so-called *torsion* and the *curvature* of any linear connection:

$$\tau = d\theta + \theta \wedge \omega, \quad \rho = d\omega + \omega \wedge \omega. \quad (6)$$

We recall the proof, in order to draw a theory which we wish both extensive and coherent. Let $\check{s} = (e_1, \dots, e_n)$ be a frame dual to s . The connection form on TM is in general $-\omega^t$, i.e. satisfies $\nabla e_i = -\sum_j e_j \omega_i^j$ or simply $\nabla \check{s} = -\check{s} s^* \omega^t$, because we require $\nabla 1 = 0$. The map $\check{s} s^* \theta^t = \sum_j e_j \theta^j s_* = \sum_j e_j e^j = 1$ is the identity endomorphism of TM . Now for any connection we have the torsion defined by $T^\nabla = d^\nabla 1$ and so we may deduce an equivariantly defined \mathbb{R}^n -valued 2-form τ on F^*M , hence vanishing on vertical directions, such that $T^\nabla = \check{s} s^* \tau^t$. We conclude $\check{s} s^* \tau^t = d^\nabla(\check{s} s^* \theta^t) = \check{s}(-s^* \omega^t \wedge s^* \theta^t + ds^* \theta^t) = \check{s} s^*(\theta \wedge \omega + d\theta)^t$. For a strictly vertical direction, hence one which cannot be represented through s , the equation follows by direct computation. Regarding the curvature equation in (6), with the above coframe we find $R^\nabla s = d^\nabla(s s^* \omega) = s s^*(\omega \wedge \omega + d\omega)$ which by definition is $R^\nabla s = s s^* \rho$, as in (3). By tensoriality, $V \lrcorner \rho = 0$ for any vertical vector field V .

We recall that θ , ω , and hence τ and ρ , are global differential forms on F^*M .

A connection is said to be *reducible* to a principal G -sub-bundle Q of F^*M , where G is a Lie subgroup of the general linear group, if $\ker \omega|_Q \subset TQ$.

Now we suppose M is also an oriented Riemannian manifold with metric $g = \langle \cdot, \cdot \rangle$. Then there is a canonical torsion-free metric connection, the Levi-Civita connection, and all the above remains true on the

principal $\mathrm{SO}(n)$ -bundle F_\circ^*M of oriented orthonormal coframes. Because it is metric, the connection matrix ω_j^i of 1-forms is skew-symmetric.

This classical theory extends to a vector bundle $X \longrightarrow M$ which is associated to a coframe principal G -bundle Q . A representation $\sigma : G \longrightarrow \mathrm{GL}(V)$ is given, where V is a vector space, so that we may write $X = Q \times_\sigma V$. This means vectors in X identify with a pair (s, f) , or any representative $(sg, \sigma(g^{-1})f)$ of their equivalence class, the obvious orbit of G , where g belongs to G . If s is a section of Q on an open set $U \subset M$ and f is any V -valued function on U , then f determines a unique function $\hat{f} : \pi|_Q^{-1}(U) \rightarrow V$ such that $f = \hat{f} \circ s$, and which satisfies $\sigma(g^{-1})\hat{f}(s) = \hat{f}(sg)$. Reciprocally, any equivariant function on Q determines a section of X . Finally, we covariantly differentiate fields of X through the class independent formula, where $\hat{\sigma} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is the induced map from σ :

$$\nabla_Z(s, f) = (s, \hat{\sigma} \cdot s^* \omega(Z)f + df(Z)) . \quad (7)$$

To prove this is well-defined on X the crucial equation to deduce first is $(sg)^* \omega = \mathrm{Ad}(g^{-1})s^* \omega + g^{-1}dg$ where g is any G -valued function defined on the domain of s .

1.2 On Riemannian 4-manifolds

Now suppose M is a connected oriented 4-manifold and let us continue with the same notation as above. First recall that $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2) / \pm 1$ and that a representation of this Lie group lies in $\Lambda^2 \mathbb{R}^4$, with kernel ± 1 , giving two similar subspaces associated to the eigenvalues of the star-operator $*$. The star or Hodge operator is invariantly defined by $\alpha \wedge * \beta = \langle \alpha, \beta \rangle \mathrm{vol}$, thus it gives an operator $*_M$ on M which, moreover, commutes with covariant differentiation. Hence we have parallel sub-bundles:

$$\Lambda^2 T^*M = \Lambda_+^2 \oplus \Lambda_-^2 . \quad (8)$$

A similar picture as the above from section 1.1 then follows for the principal $\mathrm{SO}(3)$ -bundles P_+ of norm $\sqrt{2}$ orthogonal oriented coframes of Λ_+^2 and P_- of the same type coframes of Λ_-^2 . By the last term *oriented* we just mean some choice made of one of the two connected components of the bundle of norm $\sqrt{2}$ orthogonal coframes of each of those associated vector bundles of M .

We note the spaces P_+ and P_- carry along with the $\mathrm{SO}(4)$. Choosing any oriented orthonormal coframe $e = (e^4, \dots, e^7) \in F_\circ^*M$ we then have two new coframes for the bundles of *self-dual* and *anti-self-dual* 2-forms, respectively¹:

$$e^1 = e^{45} \pm e^{67} , \quad e^2 = e^{46} \mp e^{57} , \quad e^3 = e^{47} \pm e^{56} . \quad (9)$$

This in fact determines the above choice of P_+ and P_- . Indeed, let us consider P_+ only. Any other oriented coframe of M equals $e \cdot g$ with $g \in \mathrm{SO}(4)$. The orientation of $((e \cdot g)^1, (e \cdot g)^2, (e \cdot g)^3) = (e^1, e^2, e^3) \cdot g$ is fixed by $g \in \mathrm{SO}(3)$. Since $\mathrm{SO}(4)/\mathrm{SO}(3) = S^3$ is connected, the orientation is well-defined by (9).

Now we let

$$p_+ : F_\circ^*M \longrightarrow P_+ \quad \text{and} \quad p_- : F_\circ^*M \longrightarrow P_- \quad (10)$$

¹These coframes will be useful later but in separate moments, hence we only introduce the $+$ or $-$ on the e^i , $i = 1, 2, 3$, or in other objects, when necessary. We adopt the nowadays common notation $e^{\alpha\beta} = e^\alpha \wedge e^\beta$.

be the equivariant maps defined by $p_{\pm}(e) = p_{\pm}(e^4, e^5, e^6, e^7) := (e^1, e^2, e^3)$. The kernel is $\text{SO}(3)\{\pm 1\} \simeq S^3$. The induced connections on P_{\pm} are again denoted by an $\omega = \omega_{\pm} \in \Omega^1(P_{\pm}, \mathfrak{o}(3))$, although now given by $\nabla p = p p^* \omega$ where $p = p_{\pm} \circ s$ and s is any local section of $F_{\circ}^* M \longrightarrow M$ as before and

$$\omega = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix} \quad \text{with} \quad \begin{cases} p^* \omega^1 = \omega_7^6 \pm \omega_5^4 \\ p^* \omega^2 = \omega_5^7 \mp \omega_4^6 \\ p^* \omega^3 = \omega_6^5 \pm \omega_7^4 \end{cases} . \quad (11)$$

The curvature tensor R^{Λ^2} satisfies $R_{\pm}^{\Lambda^2} p = p p^* \rho$ for a new 2-form also denoted ρ . Using the tautological form η on P_{\pm} , swiftly defined by the soldering form components within $\eta = p_{\pm}(\theta^4, \dots, \theta^7)$, and abbreviated as $\eta = (e^1, e^2, e^3)$, we easily get upon the same manifold, in due coherence with structure equations (6),

$$d\eta = \eta \wedge \omega \quad (12)$$

and again differentiating

$$0 = \eta \wedge (\omega \wedge \omega + d\omega) = \eta \wedge \rho \quad (13)$$

where ρ is the curvature 2-form

$$\rho = d\omega + \omega \wedge \omega = \begin{bmatrix} 0 & -\rho^3 & \rho^2 \\ \rho^3 & 0 & -\rho^1 \\ -\rho^2 & \rho^1 & 0 \end{bmatrix} \quad \text{with} \quad \begin{cases} \rho^1 = \rho_7^6 \pm \rho_5^4 \\ \rho^2 = \rho_5^7 \mp \rho_4^6 \\ \rho^3 = \rho_6^5 \pm \rho_7^4 \end{cases} . \quad (14)$$

One can prove by Bianchi identity (13) that the Riemann curvature tensor R^{∇} of the Riemannian 4-manifold M is symmetric, a section of $\Omega^0(S^2(\Lambda^2 T^* M))$. Let us recall the whole representation theory. Let $\{e_4, e_5, e_6, e_7\}$ be a dual frame of the above. One defines a map $\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$ by

$$\langle \mathcal{R}(e_{\alpha} \wedge e_{\beta}), e_{\gamma} \wedge e_{\delta} \rangle = -\langle R^{\nabla}(e_{\alpha}, e_{\beta})e_{\gamma}, e_{\delta} \rangle = R_{\alpha\beta\gamma\delta}^{\nabla} . \quad (15)$$

Then there are invariantly defined maps A, B, B^*, C respecting the decomposition (8), i.e. such that

$$\mathcal{R} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} . \quad (16)$$

On the other hand, of course we may write $\rho_{\pm}^i = \sum_{j=1}^3 \tilde{a}_j^i e_{\pm}^j + \tilde{b}_j^i e_{\mp}^j$. According to our conventions, using the frame $e_{\pm}^1, \dots, e_{\pm}^3$, we have $\mathcal{R}e_{\pm}^i = \sum_j \frac{1}{2} \mathcal{R}_{ij} e_{\pm}^j + \frac{1}{2} \mathcal{R}_{i\bar{j}} e_{\mp}^j$ where \mathcal{R}_{ij} follows linearly and obviously from (15). The same for $\mathcal{R}e_{\mp}^i$. With the dual frames $p_{\pm}(e_4, \dots, e_7)$ we then have $e_{\pm}^i(e_{\pm,j}) = 2\delta_j^i$, $e_{\pm}^i(e_{\mp,j}) = 0$, $\forall i, j = 1, 2, 3$, and computations with (3), (14) and $R_{\alpha\beta\gamma\delta}^{\nabla} = \rho_{\alpha}^{\beta}(e_{\gamma}, e_{\delta})$ give

$$\tilde{a}_j^i = \frac{1}{2} \rho_{\pm}^i(e_{\pm,j}) = -\frac{1}{2} \mathcal{R}_{ij} \quad \tilde{b}_j^i = \frac{1}{2} \rho_{\pm}^i(e_{\mp,j}) = -\frac{1}{2} \mathcal{R}_{i\bar{j}} . \quad (17)$$

In particular $-\tilde{a}$ is the matrix of A and $-\tilde{b}$ is the matrix of B^* . Also $\rho_{\pm}^i = \sum_{j=1}^3 \tilde{b}_j^i e_{\pm}^j + \tilde{c}_j^i e_{\mp}^j$, and again one shows $\tilde{c}_j^i = +\frac{1}{2} \mathcal{R}_{i\bar{j}}$ and the three identities $\tilde{b}_j^i = \frac{1}{2} \rho_{\mp}^i(e_{\pm,j}) = \frac{1}{2} \mathcal{R}_{i\bar{j}} = -\tilde{b}_j^i$. By (13) it is immediate that a and c are symmetric. This implies the whole symmetry of \mathcal{R} . In particular B^* is the adjoint of B .

Henceforth the curvature of the vector bundle of self-dual forms encodes half of the Riemannian curvature tensor of M . B corresponds with the traceless part of the Ricci tensor. A few lines of computation will show that M is Einstein, i.e. the so-called Ricci tensor $\text{Ric} = \sum_{\alpha=4}^7 \langle R(\cdot, e_\alpha)e_\alpha, \cdot \rangle$ is a multiple of the metric tensor, if and only if $B = 0$. In other words, M is Einstein if and only if $*\mathcal{R} = \mathcal{R}*$. If this is the case, then clearly orthogonal planes in TM have the same sectional curvature. And reciprocally.

The invariant theory of $\text{SO}(3)$ lets us define the tensors $W_+ = A - \frac{1}{3}\text{tr } A$ and $W_- = C - \frac{1}{3}\text{tr } C$, which are called the *self-dual* and *anti-self-dual Weyl* tensors of M . The so-called *Weyl* tensor $W = W_+ + W_-$ is conformally invariant, since that is certainly the case with the star operator and the W_\pm components do preserve the Λ_\pm^2 . Between those spaces we find the identities $\text{tr } A = \text{tr } C = \frac{1}{4}\text{tr}_g \text{Ric} = \frac{1}{4}\text{Scal}_M$.

The Riemannian manifold M is called *self-dual* if $W = W_+$ and called *anti-self-dual* if $W = W_-$. Clearly the former condition reads also as ($s = \frac{1}{12}\text{Scal}_M$):

$$(SD) \quad W_- = 0 \quad \Longleftrightarrow \quad \forall m \in M, \exists s \in \mathbb{R} : \rho_-^i = se_-^i + \sum_{j=1}^3 \tilde{b}_j^i e_+^j, \quad \forall i, \quad (18)$$

whereas the latter corresponds with $\rho_+^i = -se_+^i + \dots$. In any dimension, if M is Einstein, then s is known to be a constant. A reference for much of all this section is [Bes87].

1.3 G_2 -structures

These structures are well-known today and amount to a 3-form of special kind on a 7-dimensional manifold. One way to describe them is precisely within the above setting of distinguished 2-forms. Let us continue with the notation for duality from (9), but now on some oriented Euclidean 4-space, say a *horizontal* direction, which we complement with a 3-dimensional Euclidean space given by an orthonormal coframe, i.e. a set of three independent linear forms f^1, f^2, f^3 , for the *vertical* direction. Of course, along, we obtain a corresponding metric $g = g_V + g_H$ in 7 dimensions. Then a linear G_2 structure is defined on the direct sum vector space, just as in [BS89, Sal87], by

$$\phi = \lambda^3 f^{123} \mp \lambda \mu^2 (f^1 \wedge e^1 + f^2 \wedge e^2 + f^3 \wedge e^3). \quad (19)$$

In the above we read $e^i = e_\pm^i$. The coefficients are the indicated products of scalars λ, μ . A study of such forms of special type gives that the group of automorphisms of ϕ , G_2 , is a simply-connected, compact, simple, 14 dimensional Lie subgroup of $\text{SO}(7)$ where this refers to some new metric g_ϕ (cf. [Bry87]). An orientation form $o = \text{Vol}_g = f^{123} e^{4567}$ can be fixed once and for all, since connectedness prevents G_2 from jumping from one to another. The metric g_ϕ is given, for some $m \in \mathbb{R}$ yet to be determined, and for any vectors u, v , by the identity

$$u \lrcorner \phi \wedge v \lrcorner \phi \wedge \phi = \pm 6 \langle u, v \rangle_\phi m o. \quad (20)$$

In the case of (19), after some long computations for each vector e_α and f_i , giving $\|e_\alpha\|_\phi^2 = \frac{\lambda^3 \mu^6}{m}$ and $\|f_i\|_\phi^2 = \frac{\lambda^5 \mu^4}{m}$, we find

$$m^2 = \frac{1}{\|f^{123} e^{4 \dots 7}\|_\phi^2} = \frac{\lambda^{15} \mu^{12}}{m^3} \frac{\lambda^{12} \mu^{24}}{m^4}.$$

Hence the value of $m = \lambda^3 \mu^4$ and finally the metric and canonical volume form:

$$g_\phi = \lambda^2 g_V + \mu^2 g_H, \quad \text{Vol}_{g_\phi} = mo = \lambda^3 \mu^4 \text{Vol}_g. \quad (21)$$

The orientations o and mo agree if and only if $\lambda > 0$. Positive definiteness implies $\lambda, \mu > 0$. Finally the star operator $*_\phi$ for g_ϕ gives

$$\begin{cases} \phi = \lambda^3 f^{123} \mp \lambda \mu^2 (f^1 \wedge e^1 + f^2 \wedge e^2 + f^3 \wedge e^3) \\ \psi = *_\phi \phi = \mu^4 e^{4567} - \lambda^2 \mu^2 (e^1 \wedge f^{23} + e^2 \wedge f^{31} + e^3 \wedge f^{12}) \end{cases}. \quad (22)$$

Since the compatibility between the 3- and 4-dimensional subspace orientations is quite loose, but each may be previously fixed, we note one more detail. It is quite natural to start with a different set of self-dual two forms and want to keep it for some reason. For instance, say we had chosen $e^1, e^2, -e^3$ (or any other non-orientation preserving transformation). Then we can change signs to f^3 and λ . This gives the same orientation, mo , but the metric induced from the new 3-form (19) will be of signature $(3, -4)$, a so-called \tilde{G}_2 metric, where the Lie group is now the non-compact dual of G_2 . In order to have a positive definite metric we would have to start by reversing the sign in the middle of (19). For example, without further ado, we see the form $f^{123} + f^1 e_+^1 + f^2 e_+^2 - f^3 e_+^3$ is used in celebrated references such as [Bry87, Bry03, FI02, FI03, Joy09].

A G_2 structure on a 7-dimensional manifold X is given by a smooth 3-form $\phi \in \Omega_X^3$ such as that in (19) for some given frame f^1, \dots, e^7 . Then there is an induced metric g_ϕ and compatible orientation on X , as we have seen fibre-wise and for similar reasons must hold globally. The structure is furthermore reducing the holonomy of the Levi-Civita connection ∇ of this metric to G_2 if and only if $\nabla \phi = 0$. That is, an endomorphism of $T_x X$ induced by parallel displacement over a contractible loop around x is in the Lie group. Such a structure is called *parallel* or *1-flat*. A theorem of M. Fernández and A. Gray asserts this is equivalent to ϕ being harmonic.

The classification of G_2 structures is further developed in theorems due to [FG82]. It depends on four forms $\tau_i \in \Omega_X^i$ for $i = 0, 1, 2, 3$ which appear fibre-wise in $\Lambda^i T^*X$ as G_2 -modules W_i of dimensions, respectively, 1, 7, 14, 27. While the first two representation spaces W_0, W_1 are obvious, the third is $W_2 = \mathfrak{g}_2 = \{\tau_2 : \tau_2 \wedge \phi = \mp *_\phi \tau_2\}$ and the forth is $W_3 = \{\tau_3 : \tau_3 \wedge \phi = \tau_3 \wedge \psi = 0\}$. The forms indeed exist and appear in

$$\begin{cases} d\phi = \tau_0 *_\phi \phi + \frac{3}{4} \tau_1 \wedge \phi + *_\phi \tau_3 \\ d\psi = \tau_1 \wedge \psi + \tau_2 \wedge \phi \end{cases}. \quad (23)$$

Equations $d\phi = 0$ and $d *_\phi \phi = 0$, respectively, are those of a *calibrated* and *cocalibrated* G_2 structure. As we said above having both conditions is the same as $\nabla \phi = 0$. Like many authors we also reserve the name G_2 -manifold for the parallel case. If $d\phi = \tau_0 \psi$ with $\tau_0 \neq 0$ a constant, then we have a *pure type* W_0 or *nearly parallel* structure, cf. [Agr06]. For each i , the structures are called of *pure type* W_i if the only non-zero component is τ_i . Pure type W_1 is the same as locally conformally parallel, since τ_1 must be closed, i.e. locally exact.

2 The Bryant-Salamon G_2 -manifolds

2.1 Structure equations for X_+ and X_-

This section is based on [Sal87, Sal89, BS89]. We give a new description of their equations.

The manifolds $X_\pm = \Lambda_\pm^2 T^*M = P_\pm \times_{\text{SO}(3)} \mathbb{R}^3$, where the representation is the canonical, are natural vector bundles associated to a given oriented Riemannian 4-manifold M . Such manifolds carry many, rich G_2 structures. We shall treat the two $+$ and $-$ cases simultaneously, with unnecessary notation - for example in referring the 3-form ϕ . This is defined as follows assuming the notation of previous sections.

A point $x \in X_\pm$ may be written as $x = pa^t$, where $p = (e^1, e^2, e^3)$ constitutes a coframe of self- or anti-self-dual forms and $a = (a^1, a^2, a^3)$ is a vector of \mathbb{R}^3 . Then the 2-form η from (12) induces another tautological 2-form, ηa^t , well-defined on X_\pm . As well as the scalar function $r = \frac{1}{2} \|\eta a^t\|^2 = aa^t$. We have

$$d(\eta a^t) = \eta \wedge (\omega a^t + da^t) = \eta \wedge f^t \quad (24)$$

where $f = da + a\omega^t = da - a\omega$. Using either this identity or the pullback connection to X_\pm from ∇ on M , we find

$$dr = 2fa^t. \quad (25)$$

With no fear of confusion, from now on we abbreviate notation by dropping the wedge product symbol. Next we introduce a little tool to deal with several computations.

Consider the linear map which sends $\alpha \in \Omega^k(\mathbb{R}^3)$, $\forall k \geq 0$, to the $\mathfrak{o}(3)$ -valued k -form $\check{\alpha}$ exactly in the shape of the matrix $\omega = (\omega^1, \omega^2, \omega^3)^\vee$ in (11). This is,

$$\text{if } \alpha = (\alpha^1, \alpha^2, \alpha^3), \text{ then } \check{\alpha} = \alpha^\vee = \begin{bmatrix} 0 & -\alpha^3 & \alpha^2 \\ \alpha^3 & 0 & -\alpha^1 \\ -\alpha^2 & \alpha^1 & 0 \end{bmatrix}. \quad (26)$$

In coherence with our notation we also² have $\rho = \check{\rho}$. We let \cdot^\wedge denote the left inverse map, defined for any matrix A by $A^\wedge = (a_{32}, -a_{31}, a_{21})$. We have $(A^\wedge)^\vee = A$ if and only if A lies in the orthogonal Lie algebra. The following identities are trivial to check:

$$(\check{\alpha}\check{\delta})^\wedge = (\alpha^1\delta^2, -\alpha^1\delta^3, \alpha^2\delta^3) \quad \text{and} \quad (\alpha\check{\delta})^\vee = \check{\alpha}\check{\delta} - (-1)^{\deg \alpha \deg \delta} \check{\delta}\check{\alpha}. \quad (27)$$

Returning to our G_2 problem, the components $f = (f^1, f^2, f^3)$ give us the required base of 1-forms with which one defines a structure ϕ in the same fashion of (19). We define $\beta = f^{123}$ and $\text{vol} = e^{4567}$ since in fact this is the pullback to X_\pm of the volume form of M . Henceforth $\phi = \lambda^3 f^{123} \mp \lambda \mu^2 \eta f^t = \lambda^3 \beta \mp \lambda \mu^2 d(\eta a^t)$ where λ, μ are scalar functions on X_\pm . It is easy to see that

$$df = -f\omega - a\rho. \quad (28)$$

Also $\psi = \mu^4 \text{vol} - \lambda^2 \mu^2 \eta h^t$ where the 2-form h is given by $h = (\check{f}\check{f})^\wedge = (f^{23}, f^{31}, f^{12})$; note $\check{h} = \check{f}^2 = \frac{1}{2}(\check{f}\check{f})^\vee$. Applying (27) several times, we find $f\omega\check{f} = -h\omega$ and thence

$$dh = (d\check{f}\check{f} - \check{f}d\check{f})^\wedge = (df)\check{f} = -f\omega\check{f} - a\rho\check{f} = h\omega - a\rho\check{f}. \quad (29)$$

²We keep the notation for ω and ρ , the only two exceptions, everywhere referring the matrices defined earlier.

Since $\beta = \frac{1}{3}hf^t$, we have (in fact $h\omega f^t = 0$)

$$d\beta = \frac{1}{3}(dh f^t + hdf^t) = \frac{1}{3}(h\omega f^t - a\rho\check{f}f^t - h\omega f^t + h\rho a^t) = h\rho a^t \quad (30)$$

and

$$d(\eta h^t) = -\eta\check{f}\rho a^t. \quad (31)$$

Since ηf^t is exact,

$$\begin{cases} d\phi = d\lambda^3 \beta + \lambda^3 h\rho a^t \mp d(\lambda\mu^2)\eta f^t \\ d\psi = d\mu^4 \text{vol} - d(\lambda^2\mu^2)\eta h^t + \lambda^2\mu^2\eta\check{f}\rho a^t \end{cases}. \quad (32)$$

For the solution of several G_2 equations we follow [BS89, Sal87] and consider λ, μ as functions of the half square-radius r .

Proposition 2.1. *Consider the spaces $X_{\pm} = \Lambda_{\pm}^2 T^*M$ with the generic Bryant-Salamon G_2 structure ϕ . Assume λ and μ are only dependent of r . We have that $d\phi = 0$ implies the metric of M is Einstein.*

Proof. The hypothesis and (25) imply that $d\lambda = 2\frac{\partial\lambda}{\partial r}fa^t$ and analogously for μ . For the relevant part, it is now enough to see the case of self-duality. From (17), we see the Einstein condition is fulfilled with ρ_+ having no anti-self-dual terms. The $\mathfrak{o}(3)$ -valued 2-form $\rho_+ = \rho = \rho_A + \rho_B$ appears in the first line of (32). Since ϕ is closed, we must have $B = 0$. \blacksquare

In the following we find the structure forms or structure tensors according to (23).

Theorem 2.1. *Consider the spaces $X_{\pm} = \Lambda_{\pm}^2 T^*M$ with the generic Bryant-Salamon G_2 structure ϕ and assume λ and μ are only dependent of r . Assume also that M is anti-self-dual in the case of X_+ or self-dual in the case of X_- . We thus have $\rho = \mp s\eta + \rho_B$, as in equation (18), where ρ_B is the Einstein component, which interchanges self- with anti-self-duality depending of which case. Then we have:*

- i) $\tau_0 = 0$
- ii) $\tau_1 = \frac{2}{3\lambda^2\mu^4}\left(\frac{\partial(\lambda^2\mu^4)}{\partial r} - s\lambda^4\mu^2\right)dr$
- iii) $\tau_2 = \mp\left(\frac{\partial}{\partial r}\left(\frac{\mu^2}{\lambda^2}\right) - 2s\right)\left(\frac{4\lambda^3}{3\mu^2}ha^t \pm \frac{2\lambda}{3}\eta a^t\right)$
- iv) $\tau_3 = \mp\lambda^2 f\rho_B a^t$ and, in particular, $\tau_3 = 0$ if and only if M is Einstein.

Proof. i) Since the wedge of 4-forms with ϕ is equivariant, we find an invariant kernel of such map and then deduce $7\tau_0 \text{Vol}_{g_\phi} = (d\phi)\phi$. Suppose by hypothesis that $d(\lambda\mu^2) = Sfa^t = \frac{1}{2}Sdr$. Finally,

$$\begin{aligned} (d\phi)\phi &= (\lambda^3 h\rho a^t \pm S\eta f^t f a^t)(\lambda^3 \beta \mp \lambda\mu^2 \eta f^t) \\ &= s\lambda^4 \mu^2 h\eta a^t \eta f^t - S\lambda\mu^2 \eta f^t f a^t \eta f^t \\ &= 0 \end{aligned}$$

because $\rho_B \eta = 0$, $\eta f^t h\eta = \beta\eta\eta = 0$, $\eta\eta^t = \pm 6\text{vol}$, $\eta^t \eta = \pm 2\text{vol} \cdot 1_3$ and so $f\eta^t \eta f^t f = \pm 2\text{vol} f^t f = 0$ (from the structure equations we actually have $\eta\rho = 0$, but this is not quite the condition we meet with).

ii) As above, we define three functions S, T, U simply by $d(\lambda\mu^2) = Sfa^t$, $d(\lambda^2\mu^2) = Tfa^t$ and $d(\mu^4) = Ufa^t$.

Note also the identity $f\check{\eta} + \eta\check{f} = 0$, which is easy to check and implies $\eta\check{f}\check{\eta} = -f\check{\eta}^2 = \pm 4f\text{vol}$. Below we will also need $f^t f = -\check{h}$. Continuing, we have then

$$\begin{aligned} *_\phi d\psi &= *_\phi((U\text{vol} - T\eta h^t)fa^t \mp \lambda^2\mu^2\eta\check{f}s\check{\eta}a^t) \\ &= *_\phi((U - 4s\lambda^2\mu^2)\text{vol}fa^t - T\beta\eta a^t) \\ &= \frac{\lambda}{\mu^4}(U - 4s\lambda^2\mu^2)ha^t \mp \frac{1}{\lambda^3}T\eta a^t. \end{aligned}$$

Now, it is known that $\tau_1 = \frac{1}{3} *_\phi((*_\phi d\psi)\psi)$ (cf. [FG82, FI03]). Hence

$$\begin{aligned} \tau_1 &= \frac{1}{3} *_\phi(\lambda(U - 4s\lambda^2\mu^2)h\text{vol}a^t \pm \frac{T\lambda^2\mu^2}{\lambda^3}h\eta^t\eta a^t) \\ &= \frac{1}{3\lambda} *_\phi(\lambda^2U - 4s\lambda^4\mu^2 + 2T\mu^2)h\text{vol}a^t \\ &= \frac{\lambda}{3\lambda^3\mu^4}(\lambda^2U - 4s\lambda^4\mu^2 + 2\mu^2T)fa^t. \end{aligned}$$

Since $(\lambda^2U + 2\mu^2T)fa^t = \lambda^2d(\mu^4) + 2\mu^2d(\lambda^2\mu^2) = 2d(\lambda^2\mu^4)$, the result follows.

iii) The shortest path to $\tau_2(\in \mathfrak{g}_2)$ is by using the formula we have just proved. Recalling (23), we have

$$\begin{aligned} \mp *_\phi \tau_2 &= d\psi - \tau_1\psi \\ &= U\text{vol}fa^t - T\eta h^tfa^t + \lambda^2\mu^2\eta\check{f}\rho a^t - \frac{1}{3\lambda^2}(\lambda^2U - 4s\lambda^4\mu^2 + 2\mu^2T)fa^t\text{vol} + \\ &\quad + \frac{1}{3\lambda^2}(\lambda^2U - 4s\lambda^4\mu^2 + 2\mu^2T)fa^t\eta h^t \\ &= \frac{1}{3\lambda^2}(2\lambda^2U - 8s\lambda^4\mu^2 - 2\mu^2T)\eta\beta a^t + \frac{1}{3\mu^2}(\lambda^2U - 4s\lambda^4\mu^2 - \mu^2T)\eta\beta a^t \\ &= (\lambda^2U - \mu^2T - 4s\lambda^4\mu^2)(\frac{1}{3\mu^2}\eta\beta + \frac{2}{3\lambda^2}\text{vol}f)a^t. \end{aligned}$$

Hence

$$\begin{aligned} \tau_2 &= \mp(\lambda^2U - \mu^2T - 4s\lambda^4\mu^2)(\pm\frac{1}{3\mu^2\lambda^3}\eta + \frac{2}{3\lambda\mu^4}h)a^t \\ &= \mp(\frac{\lambda^2}{\mu^2}U - T - 4s\lambda^4)(\pm\frac{1}{3\lambda^3}\eta + \frac{2}{3\lambda\mu^2}h)a^t \\ &= \mp(2\frac{\lambda^2}{\mu^2}\frac{\partial\mu^4}{\partial r} - 2\frac{\partial\lambda^2\mu^2}{\partial r} - 4s\lambda^4)(\frac{2}{3\lambda\mu^2}h \pm \frac{1}{3\lambda^3}\eta)a^t \\ &= \mp(\frac{\partial}{\partial r}(\frac{\mu^2}{\lambda^2}) - 2s)(\frac{4\lambda^3}{3\mu^2}ha^t \pm \frac{2}{3}\lambda\eta a^t). \end{aligned}$$

iv) Finally, from the above τ_1 and recurring to $*_M$, the star operator lifted from M to the horizontal subspace,

we find

$$\begin{aligned}
\tau_3 &= *_\phi \left(d\phi + \frac{3}{4}\phi\tau_1 \right) \\
&= *_\phi \left(\lambda^3 h \rho a^t \pm S \eta f^t f a^t + \frac{1}{4\lambda^2 \mu^4} (\mp \lambda \mu^2 \eta f^t) (\lambda^2 U - 4s\lambda^4 \mu^2 + 2\mu^2 T) f a^t \right) \\
&= \frac{1}{4\lambda \mu^2} *_\phi \left(4\lambda^4 \mu^2 h \rho \mp 4S \lambda \mu^2 \eta \check{h} \pm (\lambda^2 U - 4s\lambda^4 \mu^2 + 2\mu^2 T) \eta \check{h} \right) a^t \\
&= \frac{1}{4\lambda \mu^2} \left(4\lambda^3 \mu^2 f *_M \rho - 4S \mu^2 \eta \check{f} + \frac{1}{\lambda} (\lambda^2 U - 4s\lambda^4 \mu^2 + 2\mu^2 T) \eta \check{f} \right) a^t \\
&= \frac{1}{4\lambda \mu^2} \left(-4s\lambda^3 \mu^2 (f \check{\eta} + \eta \check{f}) \mp 4\lambda^3 \mu^2 f \rho_B + \frac{1}{\lambda} (\lambda^2 U + 2\mu^2 T - 4\lambda \mu^2 S) \eta \check{f} \right) a^t \\
&= \mp \lambda^2 f \rho_B a^t .
\end{aligned}$$

Indeed $f \check{\eta} + \eta \check{f} = 0$ and

$$(\lambda^2 U + 2\mu^2 T - 4\lambda \mu^2 S) f a^t = 2d(\lambda^2 \mu^4) - 4\lambda \mu^2 d(\lambda \mu^2) = 0$$

So the formula becomes very simple. ■

We remark $h a^t$ is also a global 2-form, just as the 2-form ηa^t .

2.2 New examples of G_2 manifolds

With the above theorem we can construct new examples of G_2 structures of eight different and unusual classes. Regarding pure W_i , $i = 1, 2, 3$, and other relevant types, we have further observations.

One writes, in general,

$$\tau_1 = \frac{2}{3} (d \log(\lambda^2 \mu^4) - s \frac{\lambda^2}{\mu^2} dr) . \quad (33)$$

In the conditions of theorem 2.1, we can indeed find some examples of non-trivial pure type W_1 structures, i.e. locally conformally parallel. However, if $12s = \text{Scal}_M < 0$, then the structure is only locally conformally parallel, not globally, and in general the induced metric g_ϕ is not complete nor defined on the whole space. Note that s is constant since $\tau_3 = 0$. Indeed $\tau_2 = 0$ has a solution: $\lambda = \text{constant}$ and $\mu^2 = \lambda^2(2sr + c_1)$, where c_1 is another constant.

Regarding pure type W_2 structures, the equation $\tau_1 = 0$ does not yield so easily. Taking λ constant, leads to a complete solution if and only if $\text{Scal}_M \geq 0$, giving an answer to the problem. Taking μ constant, leads to another solution, but hardly with the metric g_ϕ complete.

We notice that τ_1 and τ_2 are closely related, by the following simple lemma which is just calculus in the variable r .

Lemma 2.1. *With $\lambda, \mu > 0$, any two of the following conditions imply the third:*

$$\lambda \mu = c_0 \text{ a constant} , \quad \tau_1 = 0 , \quad \tau_2 = 0 . \quad (34)$$

In order to achieve pure type W_3 or even G_2 holonomy, one thus assumes (34); equivalently the system of equations $\lambda \mu = c_0$ and $\partial_r \mu^2 - s \lambda^2 = 0$. The unique solution is (c_1 is another constant):

$$(\mu(r))^2 = (2c_0^2 s r + c_1)^{\frac{1}{2}} , \quad (\lambda(r))^2 = c_0^2 (2c_0^2 s r + c_1)^{-\frac{1}{2}} . \quad (35)$$

The only existing compact self-dual Einstein 4-manifolds with $s > 0$, result due to Hitchin, were pointed out in the original construction of what we have denoted by X_- . The following is well-known.

Theorem 2.2 (Bryant-Salamon, [BS89, Sal87]). *For $M = S^4$ and for $M = \mathbb{CP}^2$ with standard metrics, the spaces $\Lambda_-^2 T^*M$ have a complete metric with holonomy G_2 .*

We recall that self-dual (SD) scalar-flat 4-manifolds also give rise to interesting complete G_2 structures on X_- by the same method. Raising questions similar to the above for the G_2 structure on anti-self-dual (ASD) metrics, thus pretending that orientation preceeds further interests, we pass to X_+ .

Let us resume with the $\text{Scal}_M = 0$ condition. The spin compact scalar flat Kähler surfaces were classified in [LeB86, Proposition 3] and consist of the Calabi-Yau surfaces, the flat torus modulo a finite group, here denoted M_0 , and the \mathbb{CP}^1 -bundles over a Riemann surface of genus > 1 with the local product metric, here M_1 .

Theorem 2.3. *i) Let M be any complete scalar-flat Kähler surface, with the compatible orientation. Then the associated G_2 structure ϕ on the manifold X_+ is cocalibrated, i.e. $d\psi = 0$, if and only if λ, μ are constant. In this case, ϕ is of pure type W_3 and g_ϕ is complete.*

*ii) The three classes of manifolds $\Lambda_+^2 T^*K3$, where $K3$ denotes any of the homonym surfaces, $\Lambda_\pm^2 T^*M_0$, all have complete parallel G_2 structures.*

*iii) $\Lambda_+^2 T^*M_1$ is of pure type W_3 and not parallel.*

*iv) Both classes of manifolds $M_{2,k} = k\overline{\mathbb{CP}}^2$, with $k \geq 6$ (a k -many connected sum of conjugate-oriented \mathbb{CP}^2 s) and manifolds $M_{3,k} = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$, with $k \geq 14$, all with the scalar-flat ASD metrics described in [LeB04, Theorem A], admit complete G_2 structures on $\Lambda_+^2 T^*M_{i,k}$ ($i = 2, 3$) which are of pure type W_3 and not parallel.*

Proof. i) It is well-known that a Kähler surface is scalar-flat if and only if it is anti-self-dual ([Der83]), a local result. We may thus apply theorem 2.1 above to get the first part. Since we have $s = 0$, it is indeed λ and μ constant by (35), and reciprocally. Completeness follows by completeness of the totally geodesic fibres and Hopf-Rinow theorem (also cf. [Alb14, BS89]).

ii) The only spin compact cases in i) are M_0 and the $K3$ surfaces with Calabi-Yau metric ([LeB86]). Since the latter and M_0 are actually Einstein, all torsion tensors in theorem 2.1 vanish.

iii) Fibre and base of M_1 have opposite sectional curvature, but M_1 is not Einstein, so $\tau_3 \neq 0$.

iv) In [LeB04] it is shown that the metrics considered are not Einstein, so $\tau_3 \neq 0$; again taking λ, μ constant solves equations $\tau_i = 0$ for $i = 1, 2$. ■

The classification of compact simply-connected 4-manifolds with scalar-flat ASD metric consists of the $K3$ surfaces and the two classes $M_{2,k}$ and $M_{3,k}$ — the statement of LeBrun. Well understood, all the classes we have been considering are under the relation of orientation preserving isometric diffeomorphism.

Determining the holonomy subgroups of G_2 for the manifolds $\Lambda_+^2 T^*K3$ is interesting, cf. [Alb14]. The respective group for the flat class M_0 in ii) is trivial.

The next result, partly stated in [BS89], is a mirror of the Bryant-Salamon theorem 2.2, but its proof is not. First recall the complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^2 = \text{SU}(2, 1)/\text{U}(2)$, which is a ball in \mathbb{C}^2 . From [BCGP05]

we know that it is Einstein and self-dual for the canonical orientation. Let $r_0 \in \mathbb{R}^+$ and

$$D_{r_0, \pm} M = \{x \in X_{\pm} : \frac{1}{2} \|x\|^2 < r_0\} \subset \Lambda_{\pm}^2 T^* M . \quad (36)$$

Theorem 2.4. *For any given $r_0 > 0$, the real hyperbolic space $\mathcal{H}^4 = SO(4, 1)/SO(4)$ and the complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^2$, both endowed with standard metrics, are such that the disk bundle manifolds $D_{r_0, \pm} \mathcal{H}^4$ and $D_{r_0, -} \mathcal{H}_{\mathbb{C}}^2$ admit a incomplete metric with holonomy equal to G_2 .*

Proof. First, one considers of course (35) and hence may assume $c_0 = 1$. Hence $\lambda(r) = (2sr + c_1)^{-\frac{1}{4}}$ and $\mu(r) = (2sr + c_1)^{\frac{1}{4}}$ with constant $s < 0$; we recall the 3-form is $\phi = \lambda^3 \beta - \lambda \mu^2 \eta f^t$ and the metric is $g_{\phi} = \lambda^2 g_V + \mu^2 g_H$ for both of the base spaces. Since we must have $2sr + c_1 > 0$, we see that $c_1 = -2sr_0$ and we are left to play with the disk bundles. From [BCGP05] we know that $\mathcal{H}_{\mathbb{C}}^2$ is Einstein and self-dual for the canonical orientation. The incompleteness of the metric is seen by the length of a radius sitting in the disk fibres. Indeed, taking $x_0 \in X_{\pm}$ unitary for the metric on M , we have

$$\int_0^{\sqrt{2r_0}} \|tx_0\|_{\phi} dt = (-s)^{-\frac{1}{4}} \int \frac{dt}{\sqrt[4]{2r_0 - t^2}} < +\infty .$$

But a central geodesic remains in a fixed radius, hence it cannot be extended indefinitely. The holonomy equal to G_2 now follows by the results in [Alb14]. \blacksquare

Regarding vertical radial geodesics $\gamma(t) = g(t)x_0 \in D_{r_0, \pm} M$, $t \in \mathbb{R}$, they have a complicated equation, cf. [Alb14], with $s = -1$:

$$\ddot{g}(2r_0 - g^2) - \dot{g}^2 g = 0 . \quad (37)$$

We note the incompleteness of the metric is in great contrast with the elliptic geometry case. The end of the proof is accomplished with a general method found in [Alb14]. Which also gives a new proof of the $s > 0$ case, i.e. that of theorem 2.2.

Remark. It is interesting to see why, after-all, the *mirror* proof of the result about the holonomy group, from the two constant $s > 0$ base manifolds, does not work with those other two with constant $s < 0$. To guarantee the holonomy subgroup of G_2 is the whole group, [BS89] applies a general criteria which says it is sufficient that there do not exist non-trivial parallel 1-forms on the given G_2 parallel manifold. Following the article, we must first prove our manifolds Λ_{\pm}^2 are not diffeomorphic to \mathbb{R}^7 . That is true for the real hyperbolic base, a pseudo-sphere, since $\pi_3(\mathcal{H}^4) \neq 0$. But false for the complex hyperbolic ball $\overline{\mathcal{CH}}^2$ (contrary to the \mathbb{CP}^2 case). Also the proof continues with representation theory of the G -module \mathcal{P} of $\nabla^{g_{\phi}}$ -parallel 1-form fields, where G is the isometry group of the base manifold. \mathcal{P} is a vector space which is, in the real case, and should be, in the complex case, of $\dim < 7$. The isometries preserve g_{ϕ} by construction, hence G acts on \mathcal{P} . For our hyperbolic base spaces, $G = SO(4, 1)$ and $U(2, 1)$, cf. [BCGP05], which of which are the respective mirrors of the elliptic $G = SO(5)$ and $SU(3)$. We also note the orthogonal to \mathcal{P} is not finite dimensional in $\Omega_{\Lambda_{\pm}^2}^1$ so we cannot easily argue with it. A few arguments which the reader may check, valid for all cases, tell us that the G action must have irreducible components of $\dim 0, 3$ or 4 . In both elliptic cases, that is impossible and further-on implies that $\mathcal{P} = 0$. But in the real hyperbolic case there do exist representations of $SO(4, 1)$ in dimension 4, cf. [BW75].

3 G_2 structures on the frame bundle P_\pm

Given the 4-manifold M from previous sections, we consider another bundle with 3-dimensional fibres and canonical 2-forms. The principal $SO(3)$ -bundle P_\pm of norm $\sqrt{2}$ orthogonal frames of $\Lambda_\pm^2 T^*M$ described in section 1.2 can be furnished with many natural G_2 structures.

We continue to denote η, ω, ρ , respectively, the tautological 2-form field and the connection 1-form and curvature 2-form fields of skew-symmetric matrices, all three globally defined on P_\pm . They are related by $d\eta = \eta\omega$ and $\rho = d\omega + \omega\omega$ and hence also by $\eta\rho = 0$, cf. (12). One might recall these equations arise from the frame bundle of the cotangent bundle of M and its sections, which we now disregard. But indeed the resulting ω is the same for every coframe of M which induces a given \pm -dual 2-forms coframe.

Using the tools from (26), we now introduce

$$f = (\omega^1, \omega^2, \omega^3) = \hat{\omega} \quad \hat{\rho} = (\rho^1, \rho^2, \rho^3) \quad (38)$$

and again the 3-form $\beta = f^{123} = \omega^{123}$. It is very easy to find the following identities:

$$\begin{aligned} \frac{1}{2}f\omega &= (\omega^{23}, \omega^{31}, \omega^{12}) = (\omega\omega)^\wedge & \hat{\rho} &= df + \frac{1}{2}f\omega \\ \omega\hat{\rho}^t &= -\rho f^t & \beta &= \frac{1}{6}f\omega f^t & \omega f^t f &= 2\beta_{13} = f^t f\omega & \omega\omega f^t &= 0 \end{aligned} \quad (39)$$

and

$$-f\rho f^t = f\omega\hat{\rho}^t = \hat{\rho}\omega f^t = 2(\rho^1\omega^{23} + \rho^2\omega^{31} + \rho^3\omega^{12}) . \quad (40)$$

It is convenient to see further, also purely algebraic relations:

$$\begin{aligned} f\rho f^t\eta f^t &= -2(\rho^1\omega^{23} + \rho^2\omega^{31} + \rho^3\omega^{12})(e^1 f^1 + e^2 f^2 + e^3 f^3) = -2\beta\hat{\rho}\eta^t = -2\beta\eta\hat{\rho}^t \\ \eta\omega f^t\eta f^t &= f\omega\eta^t\eta f^t = \pm 2\text{vol}f\omega f^t = \pm 12\beta\text{vol} \\ \eta f^t\eta f^t &= 0 & \eta\hat{\rho}^t\eta f^t &= f\eta^t\eta\hat{\rho}^t = \pm 2\text{vol}f\hat{\rho}^t . \end{aligned} \quad (41)$$

Finally, the announced G_2 structures are the $\phi = \lambda^3\beta \mp \lambda\mu^2\eta f^t$ with positive scalar functions $\lambda, \mu \in \Omega_{P_\pm}$. Let us differentiate the components and then the forms ϕ and $\psi = *_\phi\phi = \mu^4\text{vol} - \frac{\lambda^2\mu^2}{2}\eta\omega f^t$:

$$\begin{aligned} d\beta &= \frac{1}{6}(\hat{\rho}\omega f^t - f\rho f^t + f\omega\hat{\rho}^t) = -\frac{1}{2}f\rho f^t \\ d(\eta f^t) &= \eta\omega f^t - \frac{1}{2}\eta\omega f^t + \eta\hat{\rho}^t = \eta(\frac{1}{2}\omega f^t + \hat{\rho}^t) \\ d(\eta\omega f^t) &= \eta(\omega\omega f^t + \rho f^t - \omega\omega f^t - \omega\hat{\rho}^t + \frac{1}{2}\omega\omega f^t) = -\eta\omega\hat{\rho}^t = \eta\rho f^t = 0 \\ d\phi &= d\lambda^3\beta - \frac{\lambda^3}{2}f\rho f^t \mp d(\lambda\mu^2)\eta f^t \mp \lambda\mu^2\eta(\frac{1}{2}\omega f^t + \hat{\rho}^t) \\ d\psi &= d\mu^4\text{vol} - \frac{1}{2}d(\lambda^2\mu^2)\eta\omega f^t . \end{aligned} \quad (42)$$

Now we look for the torsion tensors.

Proposition 3.1. *Let $s = \frac{\text{Scal}_M}{12}$ be the scalar function seen in (18). We then have:*

$$\tau_0 = \pm \frac{6}{7\lambda\mu^2}(\mu^2 + 2s\lambda^2) . \quad (43)$$

Proof. Recalling the equations for ρ in (16), we note the remarkable equation $\eta\hat{\rho}^t = -6\text{svol}$. With the dimensions of the vertical and horizontal 1-form subspaces in mind, we find $7\tau_0\text{Vol}_\phi =$

$$\begin{aligned}\phi d\phi &= \mp \lambda^4 \mu^2 \beta \eta \hat{\rho}^t \pm \frac{\lambda^4 \mu^2}{2} f \rho f^t \eta f^t + \frac{1}{2} \lambda^2 \mu^4 \eta \omega f^t \eta f^t \\ &= \mp \lambda^4 \mu^2 \beta (\eta \hat{\rho}^t + \hat{\rho} \eta^t) \pm 6 \lambda^2 \mu^4 \beta \text{vol} \\ &= \pm 6 \lambda^2 \mu^2 (2s \lambda^2 + \mu^2) \beta \text{vol}\end{aligned}$$

and the result follows. ■

Computations have shown it is wise to leave the hypothesis of variable μ and λ ; they considerably weight on the equations and do not seem to the author to illuminate any remarkable proposition.

Theorem 3.1. *Given any oriented Riemannian 4-manifold M , the space P_\pm admits many G_2 structures. They are defined by the above and the canonical expression $\phi = \lambda^3 \beta \mp \lambda \mu^2 \eta f^t$.*

For any constants λ, μ the G_2 structures are always cocalibrated and non-calibrated. Moreover

$$\tau_3 = \lambda^2 (*_M \hat{\rho}) f^t - \frac{1}{7} \left((\mu^2 - 12s\lambda^2) \eta f^t \mp (30s \frac{\lambda^4}{\mu^2} - 6\lambda^2) \beta \right). \quad (44)$$

Proof. The result is obtained from $\tau_3 = *_\phi d\phi - \tau_0 \phi$. The final statement is deduced next. ■

It is quite laborious to check that $\tau_3 \phi = \tau_3 \psi = 0$. One is confronted with the appearance of a 6-form $f \eta^t \hat{\rho} f^t$, which vanishes. Indeed, in the middle there is a symmetric matrix $\eta \hat{\rho}^t$, essentially the map A from (16), which we may hence diagonalize.

Recall that M is anti-self-dual (self-dual) and Einstein if in referring to P_+ (respectively, P_-) we have $\hat{\rho} = \mp s \eta$.

Corollary 3.1. *The G_2 structure ϕ is of pure type W_3 if and only if M has constant scalar curvature and μ, λ satisfy $\text{Scal}_M = -\frac{6\mu^2}{\lambda^2}$. In this case, $\tau_3 \neq 0$ since*

$$\tau_3 = \lambda^2 (*_M \hat{\rho}) f^t - \mu^2 \eta f^t \mp 3 \lambda^2 \beta. \quad (45)$$

If moreover M is also ASD (SD) and Einstein, then $\tau_3 = \pm \frac{1}{2\lambda} (\phi - 7\lambda^3 \beta)$.

Now the vanishing of τ_3 implies those curvature restrictions on duality and the Ricci tensor. The reader may also deduce the following corollary.

Corollary 3.2. *The structures (P_-, ϕ) for $M = S^4$ or \mathbb{CP}^2 , such that $s = \frac{\mu^2}{5\lambda^2}$, are nearly parallel. Moreover, $d\phi = \pm \frac{6}{5\lambda} \psi$ and hence both spaces admit G_2 structures such that $\|d\phi\|_\phi$ may be made arbitrarily small or arbitrarily large.*

We have proved above that cocalibrated G_2 structures are quite abundant. Regarding 4-dimensional geometry base, they appear naturally as, for instance, the celebrated *symplectic* cotangent bundle of every given manifold. Hence there is a chance that G_2 may lead to a new natural Hamiltonian mechanics of 4-manifolds.

Notice we could as well define ϕ through f given by any permutation of $\omega^1, \omega^2, \omega^3$. The author did not find harmonious results, as the previous, since the basic equations are then quite twisted.

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